Chapter 9

THE PALM DUALITIES IN HIGHER DIMENSIONS

1 Introduction

In the previous chapter we considered stochastic processes split into cycles by a sequence of random times (called points) and established two Palm dualities between stationary processes and cycle-stationary processes. We shall now extend this theory to $d > 1$ dimensions: to random fields 'punctuated' by a countable set of isolated points scattered over $\mathbb{R}^d$ in some random manner (a simple point process).

This extension is basically straightforward using so-called Voronoi cells instead of intervals, that is, associating to each point the set of sites that are closer to that point than to any other point. There is, however, one major complication, namely the apparent lack of a higher-dimensional analogue of cycle-stationarity. There are no cycles in higher dimensions, so what does cycle-stationarity mean there?

In one dimension cycle-stationarity means that the cycles of the process form a stationary sequence. This definition can be rephrased as *point-stationarity*: the behaviour relative to a given point is independent of the point selected as origin; the process looks the same from all the points. Note that point-stationarity is different from stationarity: stationarity means that the behaviour of the process relative to any given *nonrandom time* is independent of the time selected as origin; the process looks the same from all nonrandom times.

Point-stationarity, the property that the process looks the same from all the points, should make sense also in higher dimensions. But what does it mean, exactly? How should point-stationarity be formally defined when

$d > 1$? The answer to this question needs some motivation. Since the point-stationarity problem is what really separates the higher-dimensional case from the one-dimensional case, we shall highlight it in the structure of the chapter.

The point-stationarity problem is presented in Section 2 and solved in Section 3. After defining point-stationarity in Section 3, we further characterize the concept in Sections 4, 5, and 6. These characterizations are then used to extend to $d > 1$ dimensions the theory of the previous chapter: the point-at-zero duality is presented in Section 7 and the randomized-origin duality in Section 8. Section 9 concludes with comments on the two Palm dualities and on possible extensions of the point-stationarity concept, for instance to the zero set of Brownian motion.

2 The Point-Stationarity Problem

This section explains the point-stationarity problem in full detail, moving from the obvious one-dimensional case to the not-so-obvious higher-dimensional case. Necessary notation is introduced along the way. In order to highlight the problem we consider it first in the context of simple point processes only, not introducing the associated random field until the next section.

2.1 The Simple Point Processes N and N°

Intuitively, a simple point process in d dimensions ($d \geqslant 1$) is a countable set of isolated points scattered over the d-dimensional Euclidean space $\mathbb{R}^d$ in some random manner (like planets scattered over space). In the one-dimensional case in Chapter 8 this random set of points was written as an increasing sequence of random times. There is no natural analogue of this procedure in higher dimensions. Instead, we shall represent the random set of points (in the standard way) by a collection of random variables

$$N = (N(B) : B \in \mathcal{B}^d),$$

where $\mathcal{B}^d$ are the Borel subsets of $\mathbb{R}^d$ and

$$N(B) = \text{ the number of points in the set } B.$$

More precisely, a *simple point process* in d dimensions is a random element N in the measurable space $(M, \mathcal{M})$, where M is the set of all *simple counting measures* and $\mathcal{M}$ is the *product σ-algebra* on M, that is,

$$M = \text{set of integer-valued measures } \mu \text{ on } (\mathbb{R}^d, \mathcal{B}^d) \text{ with } \mu(B) < \infty,$$
$$\text{for all bounded } B \in \mathcal{B}^d, \text{ and } \mu(\{t\}) = 0 \text{ or } 1, \text{ for all } t \in \mathbb{R}^d,$$

Conversely, let (N°, Z°) be point-stationary under a probability measure $\mathbf{P}^\circ$ with finite Voronoi cell volume:

$$\mathbf{E}^\circ[\lambda(C_0)] < \infty.$$

Then the two stationary duals of (N°, Z°) coincide if and only if

$$\mathbf{E}[\lambda(C_0)|\mathcal{J}] = \mathbf{E}[\lambda(C_0)] \text{ a.s. } \mathbf{P}^\circ.$$

In particular, this holds in the ergodic case, that is, when $\mathbf{P}^\circ = 0$ or 1 on $\mathcal{J}$.

9.2 Random Site Change Hides the Gap Between the Dualities

As in the one-dimensional case we can make the two Palm dualities coincide by a simple random site change. Let (N, Z) be stationary under a probability measure $\mathbf{P}$ with $\mathbf{E}[1/\lambda(C_0)] < \infty$. Let Z be measurable under change of site-scale and change the site-scale by

$$G \equiv \mathbf{E}[1/\lambda(C_0)|\mathcal{J}]^{-1/d} \quad \text{to obtain} \quad (N(G\cdot), (Z_{Gs})_{s\in\mathbb{R}^d}).$$

This new pair is stationary, and its two point-stationary Palm duals coincide. This procedure *preserves* the randomized-origin duality and *not* the point-at-zero duality. In fact, we *lose* the point-at-zero duality: the point-at-zero duality merges with the randomized-origin duality by the change of site-scale and does not reappear when we return to the original site-scale after change of measure (as the randomized-origin duality does). Thus the site change is not a way to bridge the gap between the two dualities; it only *hides* it. To bridge the gap we cannot avoid a change of measure.

9.3 Extending Point-Stationarity to General Random Sets?

There are other random sets that should look the same from all the points. An obvious example is the set of times where a Brownian motion takes the value zero. The solution in the present chapter of the point-stationarity problem in the case of point processes in $d > 1$ dimensions (Definition 3.1) suggests that point-stationarity could be defined in these models as follows.

PROPOSED DEFINITION. A random set is *point-stationary* if it is distributionally invariant under bijective point-shifts against any independent stationary background.

It remains to find such backgrounds and point-shifts.

With this open problem we end the stationarity part of the book.