Chapter 8

STATIONARITY, THE PALM DUALITIES

1 Introduction

In this relatively self-contained chapter we shift the focus from coupling to stationarity. [There is, however, an obvious link to coupling because in the coupling inequalities we would like one of the processes to be a stationary version of the other. And it turns out that there are coupling applications in the end, a shift-coupling application in this chapter, exact and epsilon-coupling applications in Chapter 10.]

The aspect of stationarity under consideration here is the relation between *stationarity* and *cycle-stationarity.* A stochastic process is *stationary* if it is distributionally invariant under (nonrandom) time shifts and *cycle-stationary* if it consists of cycles forming a stationary sequence (are distributionally invariant under shifts from one cycle to the next).

We have already encountered examples of this relationship in Chapter 2. A recurrent Markov chain starting from a fixed state is split into cycles by the times of successive visits to this state. These cycles are i.i.d. and thus form a stationary sequence. In the positive recurrent case we showed (Sections 2.5 and 2.6 in Chapter 2) that in addition to this obvious cycle-stationary version the Markov chain has also a stationary version. A similar result was established for renewal processes (Section 9 of Chapter 2). A zero-delayed renewal process consists of i.i.d. intervals and thus is cycle-stationary. When the interval lengths have finite mean, we showed that in addition to this trivial cycle-stationary version the renewal process also has a stationary version.

In this chapter we consider two-sided stochastic processes split into cycles by a sequence of random times (called points) and use the simple approach of Section 9 in Chapter 2 to develop from scratch a general theory on the relation between stationarity and cycle-stationarity. The same ideas will be applied to processes in d-dimensional time (random fields) in the next chapter. The intuitive motivation for this approach is explained in the renewal case in Section 9.1 of Chapter 2.

We establish two dualities between stationarity and cycle-stationarity. In the first duality the stationary process is obtained from the cycle-stationary one by placing the origin uniformly at random in a cycle after 'length-biasing' the cycle-length. Conversely, the cycle-stationary process is obtained from the stationary one by shifting the origin to the right endpoint of the cycle straddling the origin after 'length-debiasing' the cycle-length. This duality has the following *point-at-zero* interpretation:

The cycle-stationary dual behaves like the stationary process conditioned on having a point at the origin. (1.1)

The second duality is produced in the same way as the first with the modification that the length-biasing (length-debiasing) is done under conditioning on the invariant σ-algebra. This duality has the following *randomized-origin* interpretation:

The cycle-stationary dual behaves like the stationary process with origin shifted to a uniformly chosen point; (1.2)

and conversely:

The stationary dual behaves like the cycle-stationary process with origin shifted to a time chosen uniformly in $\mathbb{R}$. (1.2°)

This is a version of so-called *Palm theory* of stationary point-processes, named after the Swedish engineer Conny Palm, who pioneered this field in the early forties. Palm theory is used, for instance, in queueing theory to derive characteristics of a queue observed at particular points (like arrival or departure instants) from the stationary characteristics, and vice versa.

In Section 2 we establish notation and present the trivial measure-free part of the dualities (shifting to and from a point), and in Section 3 we prove the key result for the change-of-measure part (length-biasing and length-debiasing). In Section 4 we present the point-at-zero duality and then motivate the point-at-zero interpretation by conditioning and limit results in Section 5, while Section 6 contains simulation applications. After introducing the invariant σ-algebra in Section 7, we present the randomized-origin duality in Section 8 and then motivate the randomized-origin interpretation by shift-coupling and Cesaro limit results in Section 9. Section 10 concludes with comments on the two Palm dualities.

2 Preliminaries – Measure-Free Part of the Dualities

In this section we shall establish the *measure-free* framework of the chapter. Although we use the words stochastic process and random times, no probability measure is present in this section.

2.1 Process and Points

Let $(\Omega, \mathcal{F})$ be a measurable space supporting

$$Z = (Z_s)_{s\in\mathbb{R}} \quad \text{and} \quad S = (S_k)_{-\infty}^{\infty}$$

where Z is a two-sided continuous-time stochastic process with a general state space $(E, \mathcal{E})$ and path space $(H, \mathcal{H})$ and S is a two-sided sequence of random times satisfying

$$-\infty \leftarrow \cdots < S_{-2} < S_{-1} < S_0 < S_1 < \cdots \to \infty$$

and

$$S_{-1} < 0 \leqslant S_0.$$

Refer to the S_n as *points*. We shall call nonrandom elements of $\mathbb{R}$ *times* (and not points) to distinguish them from these points.

Regard S as a measurable mapping from $(\Omega, \mathcal{F})$ to the *sequence space* $(L, \mathcal{L})$ where

$$L = \{(s_k)_{-\infty}^{\infty} \in \mathbb{R}^{\mathbb{Z}} : -\infty \leftarrow \cdots < s_{-1} < 0 \leqslant s_0 < s_1 < \cdots \to \infty\}$$

and $\mathcal{L}$ are the Borel subsets of L, that is,

$$\mathcal{L} = L \cap \mathcal{B}^{\mathbb{Z}}.$$

Thus the pair (Z, S) is a measurable mapping from $(\Omega, \mathcal{F})$ to $(H \times L, \mathcal{H} \otimes \mathcal{L})$. Let $\mathcal{H} \otimes \mathcal{L}^+$ denote the class of all measurable functions from $(H \times L, \mathcal{H} \otimes \mathcal{L})$ to $([0, \infty), \mathcal{B}[0, \infty))$.

2.2 The Two-Sided Joint Shift – Shift-Measurability

For $t \in \mathbb{R}$, define the (joint) *shift-map* θ_t from $H \times L$ to $H \times L$ by

$$\theta_t((z_s)_{s\in\mathbb{R}}, (s_k)_{-\infty}^{\infty}) = ((z_{t+s})_{s\in\mathbb{R}}, (s_{n_{t-}+k} - t)_{-\infty}^{\infty}), \tag{2.1}$$

where n_{t-} is determined by $(s_k)_{-\infty}^{\infty}$ as follows:

$$n_{t-} = n \quad \text{if and only if} \quad t \in (s_{n-1}, s_n]. \tag{2.2}$$

Note that θ_t is a *time* shift and shifts the points $(s_k)_{-\infty}^{\infty}$ regarded as a sequence of *times*: θ_t shifts $(s_k)_{-\infty}^{\infty}$ by subtracting t from the times s_k and only shifts the index k of $(s_k)_{-\infty}^{\infty}$ to observe the convention that zero (the time origin) lies between the points indexed by -1 and 0 [in accordance with this we call k index and not time].

In order to be able to shift at will, assume that Z is *shift-measurable*, that is, let the path set H be invariant under time shifts and the mapping taking $(z,t) \in H \times \mathbb{R}$ to $z_t \in E$ be $\mathcal{H} \otimes \mathcal{B}/\mathcal{E}$ measurable (which is equivalent to the mapping taking $(z,t) \in H \times \mathbb{R}$ to $(z_{t+s})_{s\in\mathbb{R}} \in H$ being $\mathcal{H} \otimes \mathcal{B}/\mathcal{H}$ measurable; see Section 2 of Chapter 4). Shift-measurability is all we need assume about Z in this chapter. It covers, for instance, processes with a Polish state space (in fact, separable metric suffices) and right-continuous paths. When Z is shift-measurable, then the mapping

$$\begin{aligned}&\text{taking } (((z_s)_{s\in\mathbb{R}}, (s_k)_{-\infty}^{\infty}), t) \in H \times L \times \mathbb{R}\\ &\text{to } \theta_t((z_s)_{s\in\mathbb{R}}, (s_k)_{-\infty}^{\infty}) \in H \times L\end{aligned}$$

is $\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{B}/\mathcal{H} \otimes \mathcal{L}$ measurable.

2.3 Cycles and Cycle-Lengths – Relative Position

Think of the points S as splitting Z into *cycles*

$$C_n := (Z_{S_{n-1}+s})_{s\in[0,X_n)}, \quad n \in \mathbb{Z},$$

where X_n is the nth *cycle length*,

$$X_n = S_n - S_{n-1}.$$

Thus X_0 is the length of the cycle C_0 *straddling* the origin:

$$X_0 = S_0 - S_{-1}.$$

For $t \in \mathbb{R}$, put

$$N_t = n \quad \text{if and only if} \quad t \in [S_{n-1}, S_n).$$

Note that for $s < t$, $N_t - N_s$ is the number of points in $(s,t]$, and that

$$t \geqslant 0 \quad \Rightarrow \quad N_t = \text{ number of points in } [0,t].$$

Denote the *relative position* of t in $[S_{N_t-1}, S_{N_t})$ by

$$U_t = (t - S_{N_t-1})/X_{N_t}.$$

Note that the cycle C_n is a one-sided stochastic process vanishing at the random time X_n. One way of making sense of C_n as a random element is to place it in the cemetery state Δ from time X_n onward (see Section 2.9 in Chapter 4), that is, to identify it with a one-sided stochastic process killed at time X_n,

$$C_n := \kappa_{X_n}(Z_{S_{n-1}+s})_{s\in[0,\infty)}, \quad n \in \mathbb{Z}.$$

The pair (Z,S) is determined measurably by $(S_0, (C_n)_{-\infty}^{\infty})$ and vice versa.

2.4 The Measure-Free Duality Between (Z, S) and $((Z^\circ, S^\circ), U)$

Call Z the process *associated* with S, and S the points *associated* with Z. Observe that we do not postulate any functional link between Z and S. In applications, however, S is often even determined by Z. For instance, in the Markov chain example, S is formed by the times of the successive visits of Z to a fixed state.

We shall write S° to indicate a sequence of times with a point at zero, that is,

$$S_0^\circ \equiv 0.$$

In this case we also write Z° for the associated process although the $^\circ$ does not indicate anything about the process except its association with S°.

Let U be a $(0, 1]$ valued random variable. Throughout this chapter we assume that (Z, S) and $((Z^\circ, S^\circ), U)$ are linked functionally as follows. When (Z, S) is given, define

$$(Z^\circ, S^\circ) := \theta_{S_0}(Z, S) \qquad [\text{thus } S_0^\circ \equiv 0],$$
$$U := U_{0-} \equiv -S_{-1}/X_0 \qquad [U \text{ is the } \textit{relative} \text{ position of 0 in } (S_{-1}, S_0]].$$

Conversely, when $((Z^\circ, S^\circ), U)$ is given, define

$$(Z, S) := \theta_{-(1-U)X_0^\circ}(Z^\circ, S^\circ) \qquad [\text{thus } X_0 \equiv X_0^\circ].$$

Note that (Z, S) and (Z°, S°) have the same cycles,

$$C_n \equiv C_n^\circ, \quad \text{which we can write} \quad \theta_{S_n}(Z, S) \equiv \theta_{S_n^\circ}(Z^\circ, S^\circ),$$

while $S_n \equiv (1 - U)X_0^\circ + S_n^\circ$; see Figure 2.1.

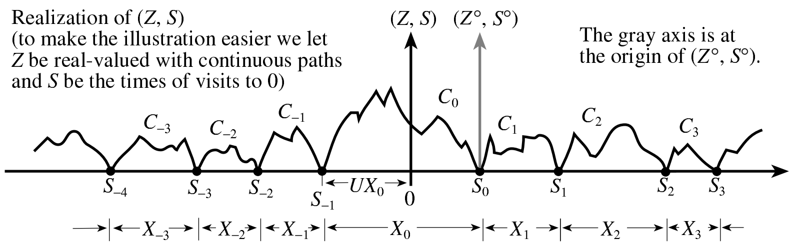

FIGURE 2.1. The functional duality between (Z, S) and $((Z^\circ, S^\circ), U)$.

3 Key Stationarity Theorem

The last section was measure-free. We now introduce a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$, that is, assume that (Z, S) is supported by the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Call (Z, S) *stationary* (under $\mathbf{P}$) if it is distributionally invariant under time shifts: under $\mathbf{P}$,

$$\theta_t(Z, S) \stackrel{D}{=} (Z, S), \quad t \in \mathbb{R}.$$

Let $\mathbf{P}^\circ$ be another probability measure on $(\Omega, \mathcal{F})$ and regard (Z°, S°) as supported by $(\Omega, \mathcal{F}, \mathbf{P}^\circ)$. Call (Z°, S°) *cycle-stationary* (under $\mathbf{P}^\circ$) if the sequence of cycles is stationary: under $\mathbf{P}^\circ$,

$$(\ldots, C_{n-1}, C_n, C_{n+1}, \ldots) \stackrel{D}{=} (\ldots, C_{-1}, C_0, C_1, \ldots), \quad n \in \mathbb{Z}.$$

Since $\theta_{S_n}(Z, S)$ is determined measurably by $(\ldots, C_{n-1}, C_n, C_{n+1}, \ldots)$ in the same way for all n, and vice versa, it follows that (Z°, S°) is cycle-stationary *if and only if*

$$\theta_{S_n}(Z, S) \stackrel{D}{=} (Z^\circ, S^\circ), \quad n \in \mathbb{Z}.$$

3.1 The Basic Equivalences

The following theorem characterizes stationarity in several ways. The link to cycle-stationarity is indicated by the last characterization, which is the key to the Palm dualities to be studied in the subsequent sections.

Theorem 3.1. *Let (Z, S) be supported by the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The following statements are equivalent:*

(*a*) *(Z, S) is stationary under $\mathbf{P}$.*

(*b*) *For $f \in \mathcal{H} \otimes \mathcal{L}^+$ and $t \in [0, \infty)$, it holds that*

$$\mathbf{E}\Big[\int_{S_0}^{S_{N_t}} f(\theta_s(Z, S))/X_{N_s}\, ds\Big] = t\mathbf{E}[f(Z, S)/X_0]. \tag{3.1}$$

(*c*) *The variable U is uniform on $(0, 1]$ and independent of (Z°, S°), and*

$$\mathbf{E}\Big[\sum_{k=1}^{N_t} f(\theta_{S_k}(Z, S))\Big] = t\mathbf{E}[f(Z^\circ, S^\circ)/X_0] \tag{3.2}$$

for $f \in \mathcal{H} \otimes \mathcal{L}^+$ and $t \in [0, \infty)$.

(*d*) *The variable U is uniform on $(0, 1]$ and independent of (Z°, S°), and*

$$\mathbf{E}[f(\theta_{S_n}(Z, S))/X_0] = \mathbf{E}[f(Z^\circ, S^\circ)/X_0] \tag{3.3}$$

for $f \in \mathcal{H} \otimes \mathcal{L}^+$ and $n \in \mathbb{Z}$.

We prove this result in the next four subsections, but let us first note several interesting consequences.

Observe first that taking $f = 1$ in (3.2) yields (since by stationarity $\mathbf{E}[N_{-t}] = -\mathbf{E}[N_t]$)

$$(Z, S) \text{ stationary} \quad \Rightarrow \quad \mathbf{E}[N_t] = \mathbf{E}[1/X_0]t, \quad t \in \mathbb{R}. \tag{3.4}$$

In particular, (3.4) yields [take $t = 1$] the following result for the *intensity* $\mathbf{E}[N_1]$ of the stationary point-stream S:

$$(Z, S) \text{ stationary} \quad \Rightarrow \quad \mathbf{E}[N_1] = \mathbf{E}[1/X_0]. \tag{3.5}$$

Also, we see from (3.4) that if $\mathbf{E}[N_1] < \infty$ then (since $\mathbf{P}(S_0 = 0) \leqslant \mathbf{E}[N_0]$) we have $\mathbf{P}(S_0 = 0) = 0$. This is in fact also true when $\mathbf{E}[N_1] = \infty$ as can be seen as follows. Let V be uniform on $[0, 1]$ and independent of S. Then, by stationarity, $\mathbf{P}(S_0 = 0) = \mathbf{P}(S_n = V \text{ for some } n)$. Since the S_n are countably many, $\mathbf{P}(S_n = V \text{ for some } n) = 0$. Thus we obtain that a stationary point-stream cannot have a point at the origin:

$$(Z, S) \text{ stationary} \quad \Rightarrow \quad \mathbf{P}(S_0 = 0) = 0.$$

Finally, due to (c), the origin of a stationary point-stream is placed uniformly at random in the cycle where it lies and independently of the process seen from one of the endpoints of the cycle:

$$(Z, S) \text{ stationary} \quad \Rightarrow \quad \begin{cases} U \text{ uniform on } (0, 1] \\ \text{and independent of } (Z^\circ, S^\circ). \end{cases}$$

This beautiful fact has the following intuitive explanation. One can think of the origin of a stationary (Z, S) as chosen uniformly at random in $\mathbb{R}$. The relative position of 0 in $(S_{-1}, S_0]$ should therefore be uniform and independent of (Z°, S°).

3.2 Proof: (a) Implies (b)

Assume that (a) holds. First suppose $f \leqslant a$. Since $N_s = 0$ for $0 \leqslant s < S_0$ and $S_0 \leqslant X_0$, we have

$$\int_0^{S_0} f(\theta_s(Z, S))/X_{N_s}\, ds \leqslant (S_0/X_0)a \leqslant a.$$