Chapter 6

MARKOV PROCESSES

1 Introduction

In this chapter we apply the three sets of coupling equivalences established in the previous two chapters (Theorem 9.4 in Chapter 4 and Theorems 5.4 and 9.4 in Chapter 5) to Markov processes. To each set of equivalences we add four more equivalent statements: on *triviality*, on *mixing*, on convergence in the *state space*, and on the constancy of *harmonic* functions.

In Section 2 we start by applying the equivalences to a single process (not necessarily Markovian) adding the triviality and mixing aspects.

Markov processes enter in Section 3, which contains preliminaries. Then each set of equivalences gets one section (Sections 4, 5, and 6) adding the two remaining aspects, on convergence in the state space and on harmonic functions. Section 7 concludes the chapter by considering the implication of these results in the case when there exists a stationary measure for the Markov process.

As in the previous two chapters we denote the time parameter by s and t in accordance with continuous time, but all we need to switch to discrete time is to restrict s and t to be integer and replace integration (over time) by summation.

2 Mixing and Triviality of a Stochastic Process

In this non-Markovian section we consider a single one-sided discrete- or continuous-time stochastic process Z with a general state space $(E, \mathcal{E})$ and some path space $(H, \mathcal{H})$. We shall apply the three sets of coupling equivalences to the two processes obtained by conditioning Z on being in two arbitrary sets of paths A and $B \in \mathcal{H}$. To each set of equivalences we add a triviality aspect and a mixing aspect: a sub-σ-algebra $\mathcal{A}$ of $\mathcal{H}$ is *trivial* with respect to Z, and Z is $\mathcal{A}$-*trivial*, if

$$\mathbf{P}(Z \in A) = 0 \text{ or } 1, \quad A \in \mathcal{A},$$

while mixing properties have to do with asymptotic independence of events happening early and events happening late in the process.

2.1 Exact coupling: $\mathcal{T}$-Triviality $\Leftrightarrow$ Mixing $\Leftrightarrow \cdots$

The word 'mixing' is used to indicate some sort of independence between what happens in a process early on and in the far future. We shall use the following definition. The process Z is *mixing* if as $t \to \infty$,

$$\sup_{A \in \mathcal{H}} |\mathbf{P}(\theta_t Z \in A, Z \in B) - \mathbf{P}(\theta_t Z \in A)\mathbf{P}(Z \in B)| \to 0, \tag{2.1}$$

for each $B \in \mathcal{H}$. Equivalently, Z is mixing if and only if (2.1) holds for all B of the finite-dimensional form

$$B = \{z \in H : z_{t_1} \in A_1, \ldots, z_{t_n} \in A_n\}, \tag{2.2}$$

where $n \geqslant 1$ and $0 \leqslant t_1 < \cdots < t_n$ and $A_1, \ldots, A_n \in \mathcal{E}$. This equivalence follows from Lemma 2.3(b) below [take $Y_t = \theta_t Z$].

Theorem 2.1. *Let Z be a one-sided discrete- or continuous-time stochastic process with a general state space $(E, \mathcal{E})$ and a general path space $(H, \mathcal{H})$. Then the following statements are equivalent.*

(a) *For each $B \in \mathcal{H}$ such that $\mathbf{P}(Z \in B) > 0$, there exists a successful distributional exact coupling of Z and the process with distribution $\mathbf{P}(Z \in \cdot | Z \in B)$.*

(b) *For each $B \in \mathcal{H}$ such that $\mathbf{P}(Z \in B) > 0$,*

$$\|\mathbf{P}(\theta_t Z \in \cdot) - \mathbf{P}(\theta_t Z \in \cdot | Z \in B)\| \to 0, \quad t \to \infty.$$

(c) *For each $B \in \mathcal{H}$ such that $\mathbf{P}(Z \in B) > 0$,*

$$\mathbf{P}(Z \in \cdot)|_{\mathcal{T}} = \mathbf{P}(Z \in \cdot | Z \in B)|_{\mathcal{T}}.$$

(d) *The process Z is $\mathcal{T}$-trivial.*

(e) *The process Z is mixing.*