Chapter 2

MARKOV CHAINS AND RANDOM WALKS

1 Introduction

We now turn to the coupling of Markov chains in discrete and continuous time, random walks, and renewal processes, the aim being to establish asymptotic properties such as asymptotic stationarity. We start with the earliest example, the classical coupling, which we present first in the pleasant context of birth and death processes.

2 Classical Coupling – Birth and Death Processes

A continuous-time irreducible nonexplosive *birth and death process* is a collection of random variables (*stochastic process*)

$$Z = (Z_s)_{s \in [0,\infty)}$$

taking values in the *state space* $E = \{0, 1, \dots\}$ and developing in time (as the *time parameter* s increases) in such a way that Z changes state only finitely often in finite time intervals (*nonexplosion*) and whenever Z visits a state i, it stays there an exponential length of time (*sojourn time*) with parameter depending only on i, and then jumps either one step up to $i+1$ (a birth) or one step down to $i-1$ (a death, this occurs only if $i > 0$) with positive probabilities depending only on i. Irreducibility follows from the positivity of the birth and death probabilities (*irreducibility* means that each state is visited with positive probability starting from any other state). Finally, we let the paths be right-continuous, that is, $Z_t = Z_{t+}$, where $Z_{t+} = \lim_{s \downarrow t} Z_s$.

2.1 Notation

Let λ be the distribution of Z_0, the *initial distribution*. Let $\mathbf{P}_\lambda$ indicate this. Let $\mathbf{P}_j$ indicate that Z starts in state j, that is, $Z_0 = j$. The semigroup of *transition matrices* is

$$P^t = (P^t_{ij} : i, j \in E), \quad t \geqslant 0, \quad [\textit{semigroup} \text{ because } P^t P^s = P^{t+s}]$$

where P^t_{ij} denotes the probability of going from i to j in a time interval of length t,

$$P^t_{ij} = \mathbf{P}(Z_{s+t} = j | Z_s = i), \quad s, t \geqslant 0, \quad i, j \in E.$$

If we treat the initial distribution λ as a row vector, then the row vector λP^t represents the distribution of Z_t,

$$\mathbf{P}_\lambda(Z_t = j) = \lambda P^t_j, \quad t \geqslant 0, \quad j \in E.$$

2.2 The Classical Coupling

Let Z' be a *differently started independent version* of Z, that is, Z' is independent of Z and has the same semigroup of transition matrices but another initial distribution λ', say. Let T be the time when Z and Z' first meet,

$$T = \inf\{t \geqslant 0 : Z_t = Z'_t\} \qquad \text{(see Figure 2.1)}.$$

Let Z'' be the process that follows the path of Z' up to T and then switches to Z,

$$Z''_t = \begin{cases} Z'_t & \text{if } t < T, \\ Z_t & \text{if } t \geqslant T. \end{cases}$$

At time T the processes Z and Z' are in the same state and will continue as if they both were starting anew in that state. Therefore, modifying Z' by switching to Z at time T does not change its distribution, that is, Z'' is a copy of Z'. Thus (Z, Z'') is a coupling of Z and Z', the *classical coupling*.

2.3 The Coupling Time T – Asymptotic Loss of Memory

The time T when Z and Z'' merge is called a *coupling time* or *coupling epoch*. By definition (Section 4.1 in Chapter 1) the event

$$\{T \leqslant t\} = \{Z_t = Z''_t\}$$

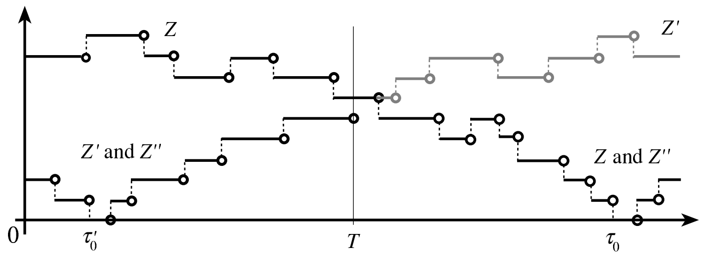

FIGURE 2.1. The classical coupling in the birth and death case.

is a coupling event for the coupling (Z_t, Z''_t) of Z_t and Z'_t. The coupling event inequality (5.11) in Chapter 1 yields the *coupling time inequality*

$$\|\lambda P^t - \lambda' P^t\| \leqslant 2\mathbf{P}(T > t), \quad t \geqslant 0. \tag{2.1}$$

The coupling is called *successful* if $\mathbf{P}(T < \infty) = 1$. This implies *asymptotic loss of memory*

$$\mathbf{P}(T < \infty) = 1 \quad \Rightarrow \quad \|\lambda P^t - \lambda' P^t\| \to 0, \quad t \to \infty. \tag{2.2}$$

2.4 Recurrence of the Birth and Death Process implies $T < \infty$

The process Z is called *recurrent* if each state $j \in E$ is *recurrent*:

$$\mathbf{P}_j(\tau_j < \infty) = 1,$$

where τ_j is the time of first visit to state j (re-entrance if Z starts in j):

$$\tau_j = \inf\{t > 0 : Z_{t-} \neq j, Z_t = j\}.$$

Recurrence implies, by irreducibility, that

$$\mathbf{P}_\lambda(\tau_j < \infty) = 1 \quad \text{for all initial distributions } \lambda \text{ and all states } j,$$

since otherwise there would be states i and j such that Z could go from j to i and never return to j, contradicting the recurrence of j.

By the birth and death property Z and Z' cannot pass each other without meeting (since jumps cannot happen simultaneously due to the exponentiality of the sojourn times in the individual states). Thus if Z starts above Z', we have $T \leqslant \tau_0$, while if Z starts below Z', we have $T \leqslant \tau'_0$, that is,

$$T \leqslant \tau_0 \vee \tau'_0 \qquad \text{(see Figure 2.1).} \tag{2.3}$$

If Z is recurrent, this implies that $\mathbf{P}(T < \infty) = 1$, and (2.2) yields that an irreducible recurrent birth and death process forgets how it started: for all initial distributions λ and λ',

$$Z \text{ recurrent} \quad \Rightarrow \quad \|\lambda P^t - \lambda' P^t\| \to 0, \quad t \to \infty. \tag{2.4}$$

Corollary 10.1. *Let S be an integer-valued renewal process with aperiodic recurrence times. Then for all integers $j \geqslant 0$ and $k \geqslant 1$,*

$$\mathbf{P}(A_n = j, B_n = k) \to \mathbf{P}(X_1 = j + k)/m, \quad n \to \infty,$$
$$\mathbf{P}(B_n = j + 1) = \mathbf{P}(A_n = j) \to \mathbf{P}(X_1 > j)/m, \quad n \to \infty.$$

The lattice version of Blackwell's renewal theorem follows immediately from the last observation by noting that

$$N\{n\} := \#\{k \geqslant 0 : S_k = n\} = 1_{\{A_n=0\}}$$

and thus $\mathbf{E}[N\{n\}] = \mathbf{P}(A_n = 0)$:

Corollary 10.2. *If S is an integer-valued renewal process with aperiodic recurrence times, then*

$$\mathbf{E}[N\{n\}] \to 1/m, \quad n \to \infty.$$

Summary

This chapter started by presenting the classical coupling (Sections 2, 3, 4). We showed that it is successful for irreducible recurrent birth and death processes and for irreducible (aperiodic in the discrete-time case) positive recurrent Markov chains. Then the Ornstein coupling was introduced (Sections 5 and 6). We showed that it is successful for integer-valued random walks with strongly aperiodic step-lengths without any recurrence condition and then applied it to construct a successful coupling of irreducible (aperiodic in the discrete-time case) null recurrent Markov chains. Finally (Sections 7 through 10), the Ornstein idea was used to construct successful epsilon-couplings of random walks with nonlattice step-lengths. When applied to renewal processes, this rendered Blackwell's renewal theorem and several other results on asymptotic stationarity.

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This ends the introductory part of the book. The next five chapters present a general coupling theory:

> Let me take you down 'cause I'm going to Strawberry Fields,
> Nothing is *real*